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THE PRESENT STATE OF MATHEMATICS.*

THE German Government has commissioned me to communicate to this Congress the assurances of its good will, and to participate in your transactions. In this official capacity, allow me to repeat here the invitation given already in the general session, to visit at some convenient time the German University exhibit in the Liberal Arts Building.

I have also the honor to lay before you a considerable number of mathematical papers, which give collectively a fairly complete account of contemporaneous mathematical activity in Germany. Reserving for the mathematical section a detailed summary of these papers, I mention here only certain points of more general interest.

When we contemplate the development of mathematics in this nineteenth century, we find something similar to what has taken place in other sciences. The famous investigators of the preceding period, Lagrange, Laplace, Gauss, were each great enough to embrace all branches of mathematics and its applications. In particular, astronomy and mathematics were in their time regarded as inseparable.

With the succeeding generation, however, the tendency to specialisation manifests itself. Not unworthy are the names of its early representatives: Abel, Jacobi, Galois and the great geometers from

* Remarks given at the opening of the Mathematical and Astronomical Congress, at Chicago, Ill.

Poncelet on, and not inconsiderable are their individual achievements. But the developing science departs at the same time more and more from its original scope and purpose and threatens to sacrifice its earlier unity and to split into diverse branches. In the same proportion the attention bestowed upon it by the general scientific public diminishes. It became almost the custom to regard modern mathematical speculation as something having no general interest or importance, and the proposal has often been made that, at least for purpose of instruction, all results be formulated from the same standpoints as in the earlier period. Such conditions were unquestionably to be regretted.

This is a picture of the past. I wish on the present occasion to state and to emphasise that in the last two decades a marked improvement from within has asserted itself in our science, with constantly increasing success.

The matter has been found simpler than was at first believed. It appears indeed that the different branches of mathematics have actually developed not in opposite, but in parallel directions, that it is possible to combine their results into certain general conceptions. Such a general conception is that of the *function*, in particular that of the analytical function of the complex variable. Another conception of perhaps the same range is that of the *Group*, which just now stands in the foreground of mathematical progress. Proceeding from this idea of groups, we learn more and more to coördinate different mathematical sciences. So, for example, geometry and the theory of numbers, which long seemed to represent antagonistic tendencies, no longer form an antithesis, but have come in many ways to appear as different aspects of one and the same theory.

This unifying tendency, originally purely theoretical, comes inevitably to extend to the applications of mathematics in other sciences, and on the other hand is sustained and reinforced in the development and extension of these latter. I assume that detailed examples of this interchange of influence may be not without various interest for the members of this general session, and on this account have selected for brief preliminary mention two of the papers which I have later to present to the mathematical Section.

The first of these papers (from Dr. Schönflies) presents a review of the progress of mathematical crystallography. Sohncke, about 1877, treated crystals as aggregates of congruent molecules of any shape whatever, regularly arranged in space. In 1884 Fedorow made further progress by admitting the hypothesis that the molecules might be in part inversely instead of directly congruent. In the light of our modern mathematical developments this problem is one of the theory of groups, and we have thus a convenient starting-point for the solution of the entire question. It is simply necessary to enumerate all discontinuous groups which are contained in the so-called chief group of space-transformations. Dr. Schönflies has thus treated the subject in a text-book (1891) while in the present paper he discusses the details of the historical development.

In the second place, I will mention a paper which has more immediate interest for astronomers, namely, a *resumé* by Dr. Burkhardt of "The Relations Between Astronomical Problems and the Theory of Linear Differential Equations." This deals with those new methods of computing perturbations, which were brought out first in your country by Newcomb and Hill; in Europe, by Gylden and others. Here the mathematician can be of use, since he is already familiar with linear differential equations and is trained in the deduction of strict proofs; but even the professional mathematician finds here much to be learned. Hill's researches involve indeed,—a fact not yet sufficiently recognised,—a distinct advance upon the current theory of linear differential equations. To be more precise, the interest centres in the representation of the integrals of a differential equation in the vicinity of an *essentially* singular point. Hill furnishes a practical solution of this problem by the aid of an instrument new to mathematical analysis,—the admissibility of which is, however, confirmed by subsequent writers,—the infinitely extended, but still convergent, determinant.

Speaking, as I do, under the influence of our Göttingen traditions, and dominated somewhat, perhaps, by the great name of *Gauss*, I may be pardoned if I characterise the tendency that has been outlined in these remarks as a *return to the general Gaussian programme*. A distinction between the present and the earlier period

lies evidently in this: that what was formerly begun by a single master-mind, we now must seek to accomplish by united efforts and coöperation. A movement in this direction was started in France some time since by the powerful influence of Poincaré. For similar purposes we three years ago founded in Germany a mathematical society, and I greet the young society in New York and its Bulletin as being in harmony with our aspirations. But our mathematicians must go further still. They must form international unions, and I trust that this present World's Congress at Chicago will be a step in that direction.

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